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DETERMINATION OF THE RELAXATION TIME OF A HEAT FLUX

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A method for determining the relaxation time of a heat flux is suggested for some nonlinear boundary-value problems of heat transfer. Modern representations of the applied theory of dynamical systems are employed.

Recently considerable attention has been focussed on the modified heat conduction equation of the hyperbolic type [1]:

$$\frac{\partial T}{\partial t} - \frac{\kappa}{\rho c_\vartheta} \frac{\partial^2 T}{\partial x^2} + \tau \frac{\partial^2 T}{\partial t^2} = 0 , \tag{0}$$

where τ is the relaxation time of the heat flux, which is comparable with the time of mean free path of the particles. Equation (0) may be derived from the classical equation

$$\rho c_\vartheta \frac{\partial T}{\partial t} = - \operatorname{div} Q \tag{1}$$

and the empirical relation

$$Q_{|t+\tau} = - \kappa \operatorname{grad} T , \tag{2}$$

where τ is the "delay" time of the response of the heat flux to a change in the temperature gradient (1).

Thus, in the zero approximation we obtain the equation

$$\tau \frac{\partial Q}{\partial t} + Q = - \kappa \operatorname{grad} T \tag{3}$$

and then equality (0) follows from (1)-(3).

Here, it is assumed (explicitly or implicitly) that $\tau > 0$ and $\ll 1$, "forgetting" that expansion in a Taylor series yields the possibility of approximating a sufficiently smooth function by polynomials. Actually,

$$Q(\cdot, t + \tau) - Q(\cdot, t) = \tau \frac{\partial Q}{\partial t}(\cdot, t) + \ldots$$

$$\ldots + \frac{\tau^n}{n!} \frac{\partial^n Q}{\partial t^n}(\cdot, t) + \frac{\tau^{n+1}}{(n+1)!} \frac{\partial^{n+1} Q}{\partial t^{n+1}}(\cdot, c) \quad (t < c < t + \tau) . \tag{4}$$

Let us assume that grad $T = 0$ in (3), and then

$$\hat{Q} = Q_0 \exp(\lambda t) \quad (\lambda = - \tau^{-1}) ,$$

where $\hat{Q}$ represents a relaxation mode and τ is in fact the relaxation time. However, what will τ be called if two terms are left in expansion (4)? Since equality (3) is the zero approximation of equality (1) – the Fourier

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hypothesis, which is fulfilled very well in some experiments – we make the following assumptions without complicating the problem:

1) zero approximation (3) is valid;

2) for all $n \geq 2$ the functions $|\partial^{n+1}Q/dt^{n+1}|$ are uniformly bounded by the common constant $M > 0$.

We now show that condition 1) is unnecessary, i.e., 1) follows from 2). In fact, from condition 2) it follows that for any $\varepsilon > 0$ $\tau > 0$ may be found such that the following estimates are obtained:

$$\frac{\tau^n}{n!} M < \frac{\varepsilon}{n}, \quad \tau^n = \frac{\varepsilon}{M}(n-1)! \quad (n \geq 2)$$

and, consequently,

$$\sum_{n \geq 2} \frac{\partial^n Q}{\partial t^n} \frac{\tau^n}{n!} \leq n \frac{\varepsilon}{n} = \varepsilon,$$

i.e., the zero approximation is valid with an accuracy to $\varepsilon > 0$.

The last estimate may be improved by assuming that

$$\left| \frac{\partial^n Q}{\partial t^n} \right| \leq M_n.$$

Usually under local equilibrium conditions $M_n \to 0$ as $n \to \infty$ and already the second derivative with respect to time must be sufficiently small, i.e., at $\tau = \varepsilon/M_2$. Then

$$Q = Q_0 \exp\left(-\frac{t}{\varepsilon} M_2\right), \quad \frac{\partial Q}{\partial t} = -Q_0 \frac{M_2}{\varepsilon} \exp\left(-\frac{t}{\varepsilon} M_2\right),$$

$$\frac{\partial^2 T}{\partial t^2} = -Q_0 \frac{M_2}{\varepsilon} \exp\left(-\frac{t}{\varepsilon} M_2\right),$$

and it is easy to see that at $\varepsilon \to 0$ the function $|\partial^2 T/\partial t^2|$ is actually small; however, tending to zero proceeds slowly: the factor τ in Eq. (0) must improve this convergence.

1. Formulation of the Boundary-Value Problem. Reduction to a System of Difference Equations. We consider system of heat transfer equations (1)-(3) and formulate the following boundary conditions to solve it:

$$Q = 0_{|x=0}, \quad Q = f(T)_{|x=l} \tag{5}$$

and some initial conditions. The linear transformation

$$Q = \frac{1}{2}(u_1 + u_2), \quad T = \frac{z}{2}(u_1 - u_2),$$

where z is some number, may reduce this problem to the form

$$\frac{\partial u_1}{\partial \bar{t}} + \frac{\partial u_1}{\partial s} = -\mu (u_1 + u_2), \tag{1'}$$

$$\frac{\partial u_2}{\partial \bar{t}} - \frac{\partial u_2}{\partial s} = -\mu (u_1 + u_2), \quad (s, \bar{t}) \in \Pi = [0, 1] \times R^+ \tag{3'}$$

with the boundary conditions

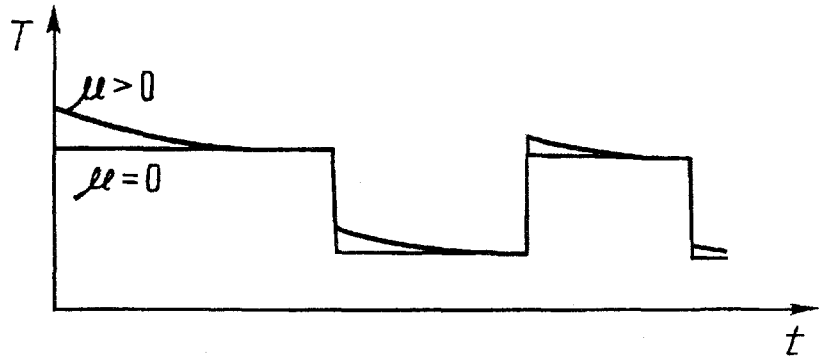

Fig. 1. Limiting temperature distribution at large times.

$$u_1 = -u_{2|s=0} \quad \text{and} \quad (-u_2) = \bar{f}(u_1)_{|s=1}; \qquad 5'$$

here $\bar{f}$ is a function prescribed implicitly by the relation

$$\frac{z}{2}(u + \bar{f}) = f\left(\frac{u - \bar{f}}{2}\right)$$

and some initial conditions: this reduction is discussed in detail in [2].

Here

$$\mu = \frac{e}{2}\sqrt{\left(\frac{\rho c_\vartheta}{\kappa\tau}\right)}, \quad z^2 = \frac{\tau}{\kappa\rho c_\vartheta}, \quad s = \frac{x}{e},$$

$$t = \left(\frac{e}{\omega}\right)\bar{t}, \quad \omega^2 = \frac{1}{z^2}.$$

We shall investigate the asymptotic behavior ($\bar{t} \to \infty$) of the solutions of this problem at extreme values of the parameter $\mu \sim \tau^{-1}$. At $\mu \to 0$ the boundary-value problem is investigated in [2], where it is shown that it is characterized by asymptotically periodic piecewise-continuous solutions that are close to piecewise-constant ones at $\mu = 0$ (Fig. 1) and take values from the set $\mathbf{P}^+$, where

$$\mathbf{P}^+ = \{\xi = \bar{f}(\xi): |\bar{f}(\xi)| < 1\}$$

is the set of attracting fixed points of the mapping $\bar{f}$ [2].

Its turns out that at $\mu \to \infty$ the situation, generally speaking, changes; however, the properties of asymptotic periodicity of the solutions are preserved in a certain sense. To illustrate this, we use the example described in detail for the Dirichlet problem in [3, p. 153]. For the system under consideration the proof, on the whole, is analogous.

We shall consider the system

$$\varepsilon(u_t + u_x) = v, \quad \varepsilon(v_t - v_x) = -v.$$

It is known [3] that by integrating each of these equations along the characteristics $\dot{x}_t = 1$ and $\dot{x}_t = -1$, respectively, at sufficiently small $\varepsilon > 0$ the solutions may be represented in the form

$$u = \varphi_1(x - t) - \varphi_2(x + t)\exp(-t/\varepsilon) + \varphi_2(x - t),$$

$$v = \varphi_2(x + t)\exp(-t/\varepsilon), \quad u = u(x, t), \quad v = v(x, t),$$

where φ_i are functions determined from the initial functions at $t = 0$, $i = 1, 2$.

We now determine the functions u and v proceeding from their values (perhaps unknown) at $x = 0$:

$$u(x, t) = \psi_1(t - x) + \psi_2(t + x) - \psi_2(t + x)\exp(-x/2\varepsilon),$$

$$v(x, t) = \psi_2(t + x)\exp(-x/2\varepsilon),$$

where ψ_1 and ψ_2 are determined from the values of u and v at the boundary $x = 0$.

Substituting these solutions into boundary conditions (2), we may obtain the following functional relations

$$\psi_1(t) + \psi_2(t) = 0,$$

$$-\psi_2(t + 1)\exp(-1/2\varepsilon) = \bar{f}(\psi_1(t - 1)) +$$

$$+\psi_2(t + 1) - \psi_2(t + 1)\exp(-1/2\varepsilon),$$

from which after shifting the argument by unity in the first equation we obtain

$$\psi_2(t + 1) = \Phi_\Lambda(\psi_2(t - 1)).$$

Assume that this equation is resolvable in such a manner that

$$\psi_2(t + 1) = \Phi_\Lambda(\psi_2(t - 1)) \quad (\Lambda = e^{-1/\varepsilon}),$$

where Φ_Λ is some (perhaps multiple-valued) function.

The asymptotic behavior of the solutions of this equation is known [2], and these solutions may be constructed by successive iterations of the initial function prescribed on the interval $[-1, 1)$.

Thus, the solutions are the asymptotically periodic piecewise-constant functions that take values from the set of attracting fixed points of the mapping Φ_Λ.

Now from the above representation for the solutions, by virtue of the constancy of the functions u and v on the characteristics $\xi = T - X$ and $\eta = T + X$, the following limiting behavior of the solutions along the characteristics may be readily obtained;

$$u \underset{t\to\infty}{\to} -\mathbf{P}_\Lambda^+ \exp(-x/2\varepsilon),$$

$$v \underset{t\to\infty}{\to} +\mathbf{P}_\Lambda^+ \exp(-x/2\varepsilon).$$

Unlike the case of small $\mu > 0$, the solutions decrease exponentially with respect to the space variable; however, the asymptotic periodicity relative to the time variable is preserved: the same occurs for the system to which the heat transfer equations are reduced.

Thus, if $\lambda = \tau^{-1}$ is interpreted as the parameter of a spectral transient mode, then the two limiting cases $\lambda = 0$ and $\lambda \to \infty$ coincide qualitatively from the viewpoint of the asymptotical behavior in time. What may occur at the intermediate values $0 < \lambda < +\infty$ is, generally speaking, unknown. However, extension to these values $\lambda \in \mathbf{R}^+$ may be achieved in the context of the "abstract" theory of dynamical systems [4].

2. Dynamical Approach. In order to understand the nature of the problem, it is sufficiently to study the difference equation with continuous time

$$u_1(t + 1) = f(u_1(t - 1)),$$

to which the heat transfer equations with nonlinear boundary conditions are ultimately reduced. Indeed, let the mapping f have the form depicted in Fig. 2. Here $a, b \in \mathbf{P}^+$, $c \in \mathbf{P}^-$, where $\mathbf{P}^-$ is the repulsive fixed point of the mapping [4]. Let $\lambda_1 = f(a)$, $\lambda_2 = f(b)$, $\lambda^- = f(c)$, where $\lambda_1, \lambda_2, \lambda^- \in \mathbf{R}^+$.

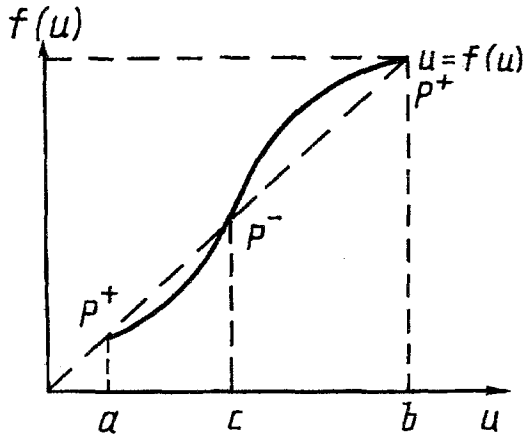

Fig. 2. "Boundary" condition determining relaxation-type fluctuations.

As is known [4], λ_1 and λ_2 characterize the rates of convergence the trajectories of solutions of the difference equation (and, consequently, of the heat transfer equation) to the equilibrium (limiting) solution while λ^- indicates the rate of divergence of the trajectories. It is also known that this tendency follows a power or exponential law [2], and for a hyperbolic system the convergence is principally the same for t (at a fixed x) and x (at a fixed t). Then from relation (1) we obtain

$$Q_0 \, e^{-\lambda t} \underset{t\to\infty}{\simeq} \kappa T_0 \, e^{-\lambda x},$$

$$-\kappa \frac{Q_0}{T_0} e^{-\lambda(t+x)} \underset{t\to\infty}{\simeq} 0$$

and therefore $\lambda = \tau^{-1}$, where τ is the relaxation time of the heat flux. Thus, we have related the definition of the time τ to the topological characteristics of the mapping f.

It is natural to choose from the two quantities λ_1 and λ_2 the smaller one: then $\lambda = \min(\lambda_1, \lambda_2)$, and consequently, $\tau = \max(\tau_1, \tau_2)$. Analogously, we may determine the increment of local instability. It is also noteworthy that $\tau > 1$ for the stable mode and $0 < \tau < 1$ for the unstable one. In the equilibrium case (see Fig. 1) it may be assumed that $\tau \to \infty$ on the "shelves" and $\tau \to 0$ at the points of discontinuity of the solution: this means that at such points the solution relaxes "instantaneously" from one equilibrium position to another.

When the mapping f is ergodic, almost all the trajectories are locally unstable, and their behavior is characterized by the instability increment

$$h_0 = \sum_{i>k}^{N} \ln |\lambda_i^-| .$$

If the mapping f additionally possesses the intermixing property ([4], p. 103), then we have an example of K-systems to which a number of problems of physics are reduced [4].

Following [4], we consider the metric invariant h of the dynamical system f. It is known that

$$h \sim h_0 \sim \tau_c^{-1},$$

where h is the entropy and τ_c is the time of splitting of correlations, and, consequently,

$$h_0 = \tau_c^{-1} = \ln \lambda^- .$$

We have dwelt on such an approach because it finds application not only in heat transfer problems but also, for instance, in laser physics [5], where, in particular, the modified Fokker–Planck equation (FPE) may be obtained for the probability density of spontaneous radiation of photons, which coincides in form with heat transfer equation (0) and follows from the hypothesis

$$W_{|t+\tau_c} = -\bar{\kappa}\,\mathrm{grad}\,P\,,$$

where P is the probability density and W is the probablity flux. The modified FPE also agrees well with the well-known Yaglom–Monin equation [6], which is obtained with account for the finite velocity of heat propagation by quite different methods.

As concerns laser physics, the methods used to determine τ_c in this field of research may be extended, almost without modification, to heat transfer problems. For instance, in laser theory the Fourier transforms of correlation functions often have a Lorentz form:

$$R(\omega) \sim \frac{1}{\pi}\,\frac{\tau_c}{1+\omega^2\tau_c^2}\,,$$

where the function R decreases rapidly for $\omega > \omega_c$ and $\omega_c = \tau_i^{-1}$ and the quantity ω_c determines the spectrum width of the system.

3. "Probability" Determination of the Relaxation Time. We now consider a somewhat different approach to the determination of the relaxation time. The following formula is well-known [7]:

$$\tau^{-1} = \sum_{\nu} P_{\nu,\nu'}\,,$$

where $P_{\nu,\nu'}$ is the probability density of particle with the wave vector k after an interaction event (e.g., creation-annihilation of a photon) into a particle with the wave vector k'. In particular, for $\nu \to \nu'$ we set that $P_{\nu,\nu'} = n^{-1}$, and then

$$\frac{1}{\tau} = n\,\frac{1}{n} = 1 \quad (\tau = 1)\,.$$

This relation "clarifies" to some extent the estimate, obtained above, for the relaxation time: $\tau \geq 1$.

We consider the Gibbs distribution over velocities for the density P [8]:

$$P(p_1, \ldots, p_N) = \frac{1}{\Phi(V, N, T)}\exp\left\{-\frac{1}{2T}\sum_{i=1}^{N}|p_i|^2\right\},$$

where

$$\Phi(V, N, T) = \int 1\,dp_1 \ldots dp_N = V^N\,,$$

here V is the phase volume occupied by N particles in velocity space. Now we identify $P_{\nu,\nu'}$ with the density $P(p_1, \ldots, p_N)$ as follows. We set

$$p_1 = \ldots = p_N = p_{cp}$$

and

$$NP_{v_{cp}v_{cp}} = P(mv_{cp}, \ldots, mv_{cp})\,.$$

Such an approximation is rough; however, it is rather realistic in the equilibrium case since combinatorial considerations may be neglected. We could introduce some combined density

$$P(v, v') = \int P(p_1, \ldots, p_N)\, dp_3 \ldots dp_N,$$

as is done in [8, p. 253]; however, this in no way influences the results formulated below. Consequently:

$$\frac{1}{\tau} = NP_{v,v'} = \frac{1}{V_{cp}^N} \exp\left\{ -\frac{N}{2T} v_{cp}^2 \right\}.$$

By definition we set $T = k_B v_{cp}^2/3$, where k_B is the Stefan–Boltzmann constant, and then

$$\tau_0 = v_{cp}^N \exp\left(m \frac{3N}{2k_B} \right). \tag{6}$$

This equality agrees with the above determinate representations of the fact that $\tau \to \infty$ in the equilibrium case. Formula (6) may be used in investigating cooperative effects (e.g., in the appearance of superradiation [9]), when the number of particles participating in the interaction must be taken into account.

One of the reasons for introducing the parameter τ is motivated, apparently, also by a desire (explicit or implicit) to take into account the local interaction of different kinds of particles because it is difficult to take into account the interaction potentials, which are often unknown. According to the Gibbs model,

$$P(x_1, \ldots, x_N, p_1, \ldots, p_N) = \frac{1}{\Phi(V, N, T)} \exp\left\{ -\frac{1}{2T} \sum_{i=1}^{N} |p_i|^2 \right\} \times$$

$$\times \exp\left\{ -\frac{1}{T} \overline{U}(x_1, \ldots, x_N) \right\}$$

and repeating of the previous calculations we obtain

$$\tau = \frac{\tau_0}{e^N} \exp\left\{ \frac{\overline{U}(x_1, \ldots, x_N)}{T} \right\},$$

where $\overline{U}(x_1, \ldots, x_N) = \sum_{i \neq j} U(x_i - x_j)$ is the potential energy. Here in the case of a model of pairwise interaction of solid spheres it may be assumed that $e = d/2$, where d is a sphere radius, but then [8]:

$$U(e) = \begin{cases} \infty, & e < d, \\ 0, & e \geq d, \end{cases}$$

and the minimum relaxation time is

$$\tau = \frac{\tau_0}{d^N}.$$

For the Ising model ([8], p. 241), we obtain

$$\tau = \frac{\tau_0}{d^N} \exp\left(\frac{a}{T} \right), \tag{7}$$

where $U(e) = a$ for $e = d$, i.e., in the case where two neighboring particles interact. Equality (7) shows that the relaxation time is proportional to the interaction "force."

4. Systems with Random Potentials. We consider the situation where, following [8], the energy of the system is random. Let, for instance,

$$dH(x, t) = \lambda dW(x, t),$$

where λ is some parameter; $W(\cdot, t)$ is a Wiener process; $x \in R^1$ is the particle coordinate. Then the generalized force in the direction of the parameter λ is equal to [8]

$$F(x, t, \lambda) = \frac{\partial H(x, t, \lambda)}{\partial \lambda} = W(x, t, \lambda),$$

where $\dot{W}(\cdot, t, \cdot)$ is a Gaussian process.

We investigate only systems whose potential energy changes with time in accordance with the law

$$dH_t = \lambda dW_t$$

and is uniformly distributed with respect to the space variable. Then the probability density of the random process H_t satisfies the Fokker–Planck equation

$$\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial H^2},$$

where D is the diffusion coefficient.

Note that for ergodic systems the coefficient D for FPE, expressed in terms of angle-action variables ([4], p. 159), is determined by some averaging at first with neglect of phase correlations:

$$D = \frac{1}{T} \langle\langle \Delta t^2 \rangle\rangle,$$

and then with account for them. Here $\tau_c << T$ is the characteristic time of change in H in K-systems (in [4, p. 160] T is considered instead of H). In [4] it is proved that

$$D = k \sum_{m=0}^{\infty} \Delta \left(\frac{1}{\tau_c}, \omega - m\Omega \right),$$

where the function

$$\pi \Delta \left(\frac{1}{\tau_c}, \omega \right), = \frac{1/\tau_c}{\omega^2 + 1/\tau_c^2}$$

is introduced such that

$$\lim_{1/\tau_c \to 0} \Delta \left(\frac{1}{\tau_c}, \omega \right) = \delta(\omega),$$

where δ is the delta-function and k is a coefficient ([4], p. 165). The term $(\omega - m\Omega)$ is of no significance for us and is determined by the specific properties of the problem for a pendulum considered in [4].

For our purposes the dependence $D = D(\omega_c)$ is of importance, which shows that the averaged diffusion coefficient depends on the correlation time. Thus, we shall introduce the correlation time for the random process H_t, but then the white-noise approximation cannot be used for the trajectory H_t:

$$\langle\langle H_{t+\tau}, H_t \rangle\rangle = \alpha \delta(\tau), \quad \alpha \in R^+,$$

and, as a consequence, the Fokker–Planck equation is not valid.

For the same reason, we adopt the hypothesis

$$W_{|t+\tau_c} = -D\frac{\partial P}{\partial H},$$

where W is the probability flux, and we write the law of conservation of probability

$$\frac{\partial P}{\partial t} = -\operatorname{div} W.$$

From these equations at $\tau_c = 0$ we obtain the FPE, while at $\tau_c > 0$ by expanding W in a Taylor series, in the zero approximaton we arrive at the system

$$\tau_c \frac{\partial W}{\partial t} + W = -D\frac{\partial P}{\partial H},$$

$$\frac{\partial P}{\partial t} + \frac{\partial W}{\partial H} = 0,$$

from which the Yaglom–Monin equations follow.

To solve this system, we formulate the following boundary conditions:

$$W = 0_{|H=0}, \quad W = \bar{f}(P)_{|H=1}$$

and some initial conditions and call such a combined problem problem S. Here $\bar{f}$ is an arbitrary nonlinear function. The hypothetical boundary condition reflects an event of some nonlinear interaction that "occurs" only when the random energy of the system attains its "maximum" value. We shall consider that this condition is of an academic character.

Similarly to the heat transfer problem it may be shown that there is a parameter

$$\bar{\mu} \sim \left(\frac{1}{D\tau_c}\right)^{1/2}$$

such that at $\bar{\mu} = 0$ problem S has asymptotically periodic piecewise-constant solutions and at $0 < \bar{\mu} << 1$ it has solutions that are close to them and take the values from $\mathbf{P}^+$, where $\mathbf{P}^+$ is the set of attracting fixed points of $\bar{f}$.

Example. Let us extend the equality obtained for determining the relaxation time to "quasi-equilibrium" solutions obtained in problem S. Then

$$\tau_t = \frac{\tau_0}{d^N}\exp\left\{\frac{H_t}{T}\right\},$$

$$E\tau_\infty = \frac{\tau_0}{d^N} E\exp\left\{\frac{H}{T}\right\} = \frac{\tau_0}{d^N}\int_0^{H_{\max}} \exp\left\{\frac{H}{T}\right\} H dH =$$

$$= \frac{\tau_0}{d^N}\mathbf{P}_1^+\int_0^{H_1}\exp\left\{\frac{H}{T}\right\}dH + \frac{\tau_0}{d^N}\mathbf{P}_2^+\int_{H_1}^{H_{\max}}\exp\left\{\frac{H}{T}\right\}dH =$$

$$= \frac{\tau_0}{d^N}\mathbf{P}_1^+ T\left\{\exp\left(\frac{H_1}{T}\right) - 1\right\} +$$

$$+ \frac{\tau_0}{d^N}\mathbf{P}_2^+ T\left\{\exp\left(\frac{H_{\max}}{T}\right) - \exp\left(\frac{H_1}{T}\right)\right\}.$$

Apparently, the last formula could be used to determine the temperature in a crystal lattice with an interaction potential that is random in time and uniform in space. If with an account for the equations themselves we write the boundary condition at $H = 1$ in the form

$$- D \frac{\partial P}{\partial H} = \bar{f}(P),$$

then for a quantum solid the probability P may be considered to be proportional to the square of the amplitude of the energy of elementary excitations in the lattice and $\partial P/\partial H$ may be interpreted as the amplitude shift relative to the energies in the vicinity of lattice oscillations.

NOTATION

T, temperature; Q, heat flux; τ, relaxation time of a heat flux; k, thermal conductivity; ρ, density; c_v, specific heat; k_B, Stefan–Boltzmann constant.

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